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# String networks as tropical curves

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Abstract: A prescription for obtaining supergravity solutions for planar $(p, q)$-string networks is presented, based on earlier results. It shows that networks may be looked upon as tropical curves emerging as the spine of the amoeba of a holomorphic curve in M-theory. The Kähler potential of supergravity is identified with the corresponding Ronkin function. Implications of this identification in counting dyons is discussed.

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String or brane networks are configurations in string theory made up of intersecting strings or branes, preserving lesser supersymmetry compared to a single brane. Networks of electrically as well as magnetically charged $(p, q)$-strings, in particular, have had a crucial role in the understanding of duality symmetries of string theory since the early days of M- and F-theories [1 − 5]. Following the identification of networks of strings or branes as dyonic states in string theory, which in turn are related by duality symmetries to black holes, the counting of such networks have recently become important for a microscopic understanding of black hole entropy from string theory [6 − 11].

String or brane networks may be studied within different frameworks, namely, the world-sheet description of string theories, the world-volume theories of D-branes or supergravity. String networks have further been studied in M-theory [12].

In this note we present a prescription for obtaining a supergravity solution for general $(p, q)$-string networks, directly related to their M-theoretic description as wrapped membranes. The outline of the procedure is as follows. A $(p, q)$-string of type-IIB string theory is described as a membrane in M-theory with one circle wrapped on a torus. A network, in this setting, is characterized by a holomorphic curve, called a spectral curve or a brane profile, written in suitable coordinates [13, 12]. The solution proposed here hinges on the observation that a precise description of planar string networks may be given as a tropical curve [14 − 16] corresponding to the spectral curve. The asymptotic $(p, q)$ charges that characterize a network are given as the degree of the tropical curve [17]. The tropical curve is obtained as the spine of the amoeba of the spectral curve [18]. The nexus between the M-theoretic and the tropical descriptions provides an inkling of the choice of the Kähler potential in the eleven-dimensional supergravity. We identify the Ronkin function of the amoeba [14 − 16, 18] as the sole contribution to the Kähler potential due to the network, thereby drastically simplifying the corresponding Monge-Ampère equation. This leads to an explicit solution for *any* planar $(p, q)$-string network and relates the membrane description of networks with the supergravity description in eleven dimensions.

The identification of string networks as tropical curves has interesting implications for their counting, which in turn yields the degeneracy of certain 1/4-BPS dyons [6 − 11]. Tropical curves are duals to subdivisions of Newton polygons [14]. Thus the number of tropical curves of a given degree is related to the number of regular subdivisions of the corresponding Newton polygon. While so-called singular tropical curves may be conceived, corresponding to non-regular subdivisions, they will not correspond to configurations with three-string junctions only. Henceforth a tropical curve will refer to a non-singular one. Thus, the degeneracy of $(p, q)$-string networks, for a given set of asymptotic charges, receives a combinatorial description under the above-mentioned identification.

We shall begin with a brief discussion of tropical curves and their Ronkin function. We then identify a planar string network in its membrane avatar as a tropical curve. Relating the Kähler potential of the eleven-dimensional supergravity to the Ronkin function of the spectral curve then leads to a very simple, but general, explicit solution. We close with a discussion of applications to the counting problem.

Let us recall the description of the amoeba of a complex curve and its spine [14]. Let us consider a curve $\mathcal{C}$ in the affine space $\mathbf{C}^2$, with coordinates $(u^1, u^2)$, given by a polynomial equation

$$\mathcal{C} = \left\{ (u^1, u^2) | f(u^1, u^2) = \sum_{i,j\in\mathbf{N}} a_{ij} (u^1)^i (u^2)^j = 0 \right\}, \tag{1}$$

where $a_{ij}$ are complex coefficients and $\mathbf{N}$ denotes the set of natural numbers. The curve $\mathcal{C}$ is first restricted to $(\mathbf{C}^\star)^2$, where $\mathbf{C}^\star$ denotes the complex plane sans the origin. The restricted set is mapped, in turn, to the real plane by the Log-map,

$$\begin{aligned} \mathrm{Log} : (\mathbf{C}^\star)^2 &\longrightarrow \mathbf{R}^2 \\ u = (u^1, u^2) &\longmapsto (x^1, x^2) := (\log|u^1|, \log|u^2|). \end{aligned} \tag{2}$$

The resulting subset $\mathcal{A}_\mathcal{C} = \mathrm{Log}(\mathcal{C} \cap (\mathbf{C}^\star)^2)$ of $\mathbf{R}^2$ is called the *amoeba* of the curve $\mathcal{C}$. A family of amoebas parametrized by a small real number $\zeta$ is obtained by considering the Log-map with base $\zeta$ as

$$\begin{aligned} \mathrm{Log}_\zeta : (\mathbf{C}^\star)^2 &\longrightarrow \mathbf{R}^2 \\ u = (u^1, u^2) &\longmapsto (x^1, x^2) := (-\log_\zeta|u^1|, -\log_\zeta|u^2|). \end{aligned} \tag{3}$$

The definition (2) corresponds to $\zeta = 1/e$. Reducing the family parameter results in shrinking the amoeba. In the limit of vanishing $\zeta$ we obtain the spine of the amoeba [14, 18], called the *tropical curve*, denoted $\mathcal{C}_T$, corresponding to the curve $\mathcal{C}$.

While this analytic definition is intuitively appealing, it is often easier to enumerate tropical curves using a more combinatorially amenable algebraic definition. Given an algebraic curve $\mathcal{C}$ as in (1) one first defines the function [1],

$$g(x^1, x^2) := \max\{ix^1 + jx^2 - \mathrm{val}(a_{ij}), (i,j) \in \mathbf{N}^2, a_{ij} \neq 0\}, \tag{4}$$

where val, called the valuation, is an indicial weight assigned to the coefficients. The tropical curve $\mathcal{C}_T$ is the corner locus of this convex piecewise linear function, that is the locus of points at which $g$ is not differentiable. Given a complex curve $\mathcal{C}$, both the definitions yield the same tropical curve.

Let us illustrate the definitions with the simplest example of the curve

$$\mathcal{C} = \{(u^1, u^2) \in \mathbf{C}^2 | u^1 + u^2 = 1\}. \tag{5}$$

The amoeba of this curve obtained from (2) is plotted in figure 1(a). The tropical curve is obtained from the combinatorial definition. The function $g$ becomes

$$g(x^1, x^2) = \max(x^1, x^2, 0), \tag{6}$$

yielding three line segments $\{x^1 = 0, x^2 < 0\}$, $\{x^1 < 0, x^2 = 0\}$, $\{x^1 = x^2 > 0\}$. The corresponding tropical curve is the tree shown in figure 1(b). It clearly is the limit as the amoeba in figure 1(a) shrinks to its spine. The between the tree and a basic three-string junction consisting of strings of charges $(0, -1)$, $(-1, 0)$ and $(1, 1)$ is conspicuous

[1] Formally, the coefficients $a_{ij}$ appearing in (1) are taken to be valued in a formal power-series, the so-called Puiseux series [14]. However, this difference will be inconsequential for our purposes here.



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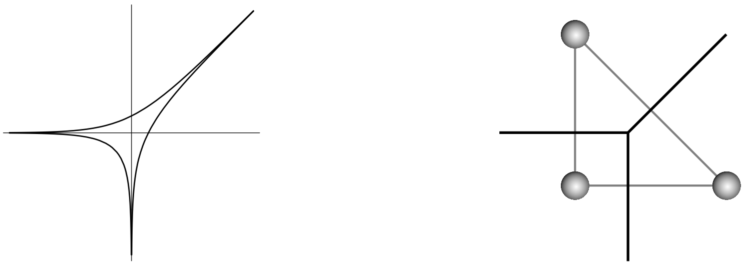

(a) Amoeba of the curve $u^1 + u^2 = 1$ (b) Tropical curve and Newton polygon

**Figure 1:** Amoeba and tropical curve

(a) (b)

**Figure 2:** Tropical curves corresponding to different subdivisions of a Newton polygon

from figure 1(b). Indeed, as we shall note below, planar string networks constructed from $(p, q)$-strings can be defined as tropical trees in general.

We can associate a Newton polygon to the curve $\mathcal{C}$ as the convex hull of the lattice points $(i, j)$ in $\mathbf{R}^2$ appearing in (1). The tropical curve is obtained as the dual, that is by drawing line segments in the plane perpendicular to the lines in a regular subdivision of the Newton polygon. For the above example, the Newton polygon has the lattice points $(0, 0)$, $(0, 1)$ and $(1, 0)$ as its vertices, as shown in figure 1(b). Different regular subdivisions of the Newton polygon yields different tropical curves, with the same degree (number of external legs) as illustrated in figure 2, where tropical curves corresponding to the most general quadratic plane curve are shown.

Finally, let us recall that the Ronkin function associated to the curve $\mathcal{C}$, or, equivalently, to the amoeba $\mathcal{A}_{\mathcal{C}}$, is defined as [20]

$$N_f(x^1, x^2) = \frac{1}{(2\pi i)^2} \int\limits_{\substack{|u^1|=e^{x^1} \\ |u^2|=e^{x^2}}} \log f(u_1, u_2) \frac{du^1}{u^1} \frac{du^2}{u^2} \tag{7}$$

The Ronkin function is convex over the amoeba and linear over each component of its complement [16]. The Ronkin function of a Harnack curve [19], which we assume of the curves that we consider here, thereby guaranteeing maximality of the area of the amoeba in the Lebesgue measure, satisfies the Monge-Ampère equation [20],

$$\frac{\partial^2 N_f}{\partial x^1 \partial x^1}\frac{\partial^2 N_f}{\partial x^2 \partial x^2} - \frac{\partial^2 N_f}{\partial x^1 \partial x^2}\frac{\partial^2 N_f}{\partial x^2 \partial x^1} = \frac{1}{\pi^2}. \tag{8}$$

Let us now go over to the description of string networks from membranes in M-theory [12], envisaged as tropical curves. We shall consider the eleven-dimensional supergravity limit of M-theory on $\mathbf{R}^{1,8} \times T^2$, where the torus $T^2$ is parametrized by the coordinates $x^3$ and $x^{10}$, with periodicities[2]

$$(x^3, x^{10}) \sim (x^3 + 2\pi R, x^{10}) \sim (x^3 + 2\pi R, x^{10} + 2\pi R). \tag{9}$$

For a finite $R$ this describes type-IIB theory on a circle, while in the limit of vanishing $R$ one recovers the type-IIB theory in ten dimensions. A network of $(p, q)$-string lying in the $(x^1, x^2)$-plane is described in this setting through an auxiliary curve, holomorphic, in order to be supersymmetric. We can define complex coordinates

$$z^1 = x^1 + ix^3, \quad z^2 = x^2 + ix^{10}. \tag{10}$$

parametrizing $\mathbf{C}^2$. Then the coordinates

$$u^1 = e^{z^1/R}, \quad u^2 = e^{-z^2/R}. \tag{11}$$

parametrize $(\mathbf{C}^\star)^2$. A single $(p, q)$-string, lying in the $(x^1, x^2)$-plane is specified by a holomorphic curve in the $(u^1, u^2)$ coordinates as

$$(u^1)^p (u^2)^q = 1. \tag{12}$$

The string itself may be obtained from this as the tropical limit

$$px^1 = qx^2, \tag{13}$$

following the algebraic definition. More generally, a string network is specified by giving a spectral curve [12]

$$f(u^1, u^2) = \sum_{p,q \in \mathbf{Z}} (u^1)^p (u^2)^q = 0. \tag{14}$$

The spectral curve describes a membrane that wraps on the torus $T^2$ yielding the string network in the tropical limit. The description of networks as tropical curves coincides with the more traditional picture obtained by analyzing the asymptotics [12].

Let us now consider the supergravity description of string networks. Different aspects of supergravity configurations corresponding to networks of strings and branes have been extensively studied [13, and reference therein]. Here we only quote a few formulas relevant for the case at hand, namely, the M-theoretic description of planar string networks at low

[2]The convention for periodicities of the torus coordinates are chosen after [13] and differ from the one in [12]. But this is only a change of coordinates, resulting in a change of basis of the homology cycles of $T^2$, which does not affect the physical properties. The string coupling is also set to unity.

energies. We seek a metric as well as associated fluxes corresponding to a string network in the eleven-dimensional supergravity. Solutions in type-IIB string theory ensues in the limit of vanishing $R$. upon assuming isometries along two directions $x^3$ and $x^{10}$. The metric ansätz for the eleven-dimensional geometry, corresponding to an M-theory configuration with $\mathrm{U}(1)_t \times \mathrm{SO}(6)$ symmetry and preserving eight supercharges is [13]

$$\begin{aligned} ds^2 &= -e^{2A}dt^2 + 2e^{2A}h_{a\bar{b}}dz^a d\bar{z}^b + e^{-A}(dy^2 + y^2 d\Omega_5^2), &(15)\\ h_{a\bar{b}} &= \frac{\partial^2 K}{\partial z^a \partial \bar{z}^b}, &(16)\end{aligned}$$

where $K$ denotes the Kähler potential. The string lies within a four-dimensional subspace of the eleven-dimensional space-time, with complex coordinates as in (10), where $x^{10}$ denotes the coordinate of the eleventh dimension of M-theory. Also, $y$ and $\Omega_5$ denote, respectively, the radial and angular coordinates of the six-dimensional part of the space-time transverse to $\mathbf{C}^2(z^1, z^2)$, while $t$ denotes the temporal coordinate. The fluxes are determined in terms of the function $A$ appearing in the metric and the Kähler potential. In order for a network lying in the $x^1$-$x^2$-plane to preserve some supersymmetry, the Kähler potential $K$ is required to satisfy [13]

$$\frac{\partial^2 K}{\partial z^1 \partial \bar{z}^1}\frac{\partial^2 K}{\partial z^2 \partial \bar{z}^2} - \frac{\partial^2 K}{\partial z^1 \partial \bar{z}^2}\frac{\partial^2 K}{\partial z^2 \partial \bar{z}^1} = \frac{1}{4}e^{-3A}, \tag{17}$$

$$\nabla_y K = -2e^{-3A}, \tag{18}$$

where $\nabla_y$ denotes the $y$-Laplacian.

Assuming a further $\mathrm{U}(1)^2$ isometry, corresponding to periodicities along the coordinates $x^3$ and $x^{10}$ of the torus $T^2$, (17) reduces to

$$\frac{\partial^2 K}{\partial x^1 \partial x^1}\frac{\partial^2 K}{\partial x^2 \partial x^2} - \frac{\partial^2 K}{\partial x^1 \partial x^2}\frac{\partial^2 K}{\partial x^2 \partial x^1} = \frac{1}{4}e^{-3A}. \tag{19}$$

Given a particular network, a solution to this determines its effect on the geometry of the target space. A simple solution arises by comparing (19) and (8). We write the Kähler potential using the Ronkin function corresponding to the spectral curve $\mathcal{C}$ as

$$K = N_f(x^1/R, -x^2/R) + \psi(y), \tag{20}$$

where now $R$ is a function of $y$ only and so is $\psi$, thanks to the SO(6) symmetry. Moreover, $R(y)$ is stipulated to vanish at a single value of $y$ signalling the presence of a source. Using the expressions $|u^1| = e^{x^1/R}$ and $|u^2| = e^{-x^2/R}$, we note that the Kähler potential satisfies,

$$\frac{\partial^2 K}{\partial x^1 \partial x^1}\frac{\partial^2 K}{\partial x^2 \partial x^2} - \frac{\partial^2 K}{\partial x^1 \partial x^2}\frac{\partial^2 K}{\partial x^2 \partial x^1} = \frac{1}{\pi^2 R^4}, \tag{21}$$

in view of (8). This, in turn, constrains the function $A$ to be a function of $y$ alone, by (18), such that $e^{-3A} = 4/\pi^2 R^4$ while the function $\psi(y)$ now satisfies the six-dimensional Laplace equation

$$\nabla_y K = -\frac{8}{\pi^2 R^4}. \tag{22}$$

Intuitively, in the presence of network sources the geometry is expected to be shaped after the amoeba. The identification of string networks as tropical curves and writing the Kähler potential in terms of the Ronkin function realizes this, yielding a simple solution to *all* planar string networks. The Kähler potential and hence the metric $h_{a\bar{b}}$ depends on the specific network chosen as the Ronkin function is associated to the spectral curve. The network affects the transverse part of the space-time through $R$. The $y$-dependence of the Kähler function is not fixed at this stage, due to the provision of adding an arbitrary function of $y$ to the Kähler potential [13]. Fixing it requires imposing specific boundary conditions. The amoeba of the curve $\mathcal{C}$ goes over to the string network in the limit of vanishing $R$, if we identify the parameter $\zeta$ as $\zeta = e^{-1/R}$.

Let us now briefly indicate some consequences of these considerations. As illustrated in figure 2, string networks with specified asymptotic charges, looked upon as tropical curves with a specified degree corresponding to external legs, may differ in the internal structure. Two different networks may thus correspond to the same spectral curve as the spectral curve only determines the Newton polygon, and not its triangulation. A Newton polygon admits, more often than not, various subdivisions. Hence more than one tropical curves, all with the same degree (external legs), correspond to the same Newton polygon by duality, mentioned earlier. Translated to networks, this implies that there is a degeneracy of string networks with specified asymptotic charges, corresponding to a spectral curve. With our identification of networks as tropical curves, now, the degeneracy equals the number of regular subdivisions of the Newton polygon corresponding to the spectral curve. While there seems to be no general formula for the number of subdivisions of polygons, some estimates exist [21, 22], especially for polygons of small size, as well as other numerical means, which are now at our disposal. As a simple application let us note that an $\mathrm{SL}(2, Z)$ transformation does not change the volume of the Newton polygon of figure 1(b). Hence under this transformation the Newton pollygon does not pick up any extra lattice point. It follows that the number of subdivisions does not change and we conclude that an S-duality transformation of the basic three-string junction shown in figure 1(b) does not alter the degeneracy [11]. On the other hand, the generic spectral curve corresponding to the configurations in figure 2 is a quadric. This network does not belong to the S-duality orbit of the basic junction of figure 1(b), [11]. Hence the known string theoretic formulas can not be used to calculate its degeneracy. However, the number of regular subdivisions of the corresponding Newton polygon is two, predicting a doubly degenerate string network.

To conclude, in this note we have presented a simple prescription for obtaining supergravity solutions of planar $(p, q)$-string networks. It incorporates the M-theoretic membrane description of the networks in terms of spectral curves. A network is defined as the spine of the amoeba of the spectral curve. The verisimilitude of string networks and grid diagrams with tropical curves have been noted earlier [17, 23 − 27]. However, obtaining a supergravity solution ensuing from identifying them establishes a precise connection. The prescription works for all planar networks. Furthermore, the networks are "sensed" by the transverse coordinates solely by the presence of a source as a singularity in (22). The solution is generic in this sense. Defining networks as tropical curves have important consequences in estimating the degeneracy of networks combinatorially. We hope that this will be useful in studying the degeneracies in general. Generalizations to higher dimensional networks [4, 5, 13] appears possible and will be reported separately.

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